Superposition of chaotic process with convergence to Lévy's stable law

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Abstract

We construct a family of chaotic dynamical systems with explicit broad distributions, which always violate the central limit theorem. In particular, we show that the superposition of many statistically independent, identically distributed random variables obeying such chaotic process converge in density to Lévy's stable laws in a full range of the index parameters. The theory related to the connection between deterministic chaos and non-Gaussian distributions gives us a systematic view of the purely mechanical generation of Lévy's stable laws.

The central limit theorem (CLT) breaks down when any stationary stochastic process has infinite variance. Thus, it is an important question whether this mathematically pathological situation is physically relevant or not. One noteworthy point arising from recent studies is that such examples of Lévy's stable laws, which are the most famous class of distributions violating the CLT, can be seen in many different fields, such as astronomy, physics, biology, economics and communication engineering, under broad conditions [1–4]. Thus, it is our primary interest to determine why such Lévy's stable laws are widely observed, and furthermore, to elucidate the mechanism of generating Lévy's stable laws. In the 1980's, there were several studies which clarified the relation between intermittent periodic mapping and anomalous diffusion with Lévy's law-like broad distributions [5,6]. Random-walk models [7] and combinations of several random number generators [8,9] are also utilized to generate Lévy's stable laws. However, such analyses include approximations or the non-deterministic nature in the models themselves, or else their generation methods are only applicable to a special class of Lévy's stable laws. The purpose of the present paper is to present a systematic method for exact and purely mechanical generation of stable laws with arbitrary indices only using concrete chaotic dynamical systems. Let us consider an one-dimensional dynamical system

$$X_{n+1} = \frac{1}{2}(X_n - \frac{1}{X_n}) \equiv f(X_n) \tag{1}$$

on the infinite support $(-\infty, +\infty)$. Note that this mapping $f(X)$ can be seen as the doubling formula of $-\cot(\theta)$ as $-\cot(2\theta) = f[-\cot(\theta)]$. Thus, the system has the exact solution $X_n = -\cot(\frac{\pi}{2}2^n\theta_0)$. Using a diffeomorphism $x \equiv \phi^{-1}(\theta) = -\frac{1}{\tan(\frac{\pi}{2}\theta)}$ of $\theta \in [0,2]$ into $]-\infty, +\infty[$, we derive the piecewise-linear map $g^{(2)}(\theta) = \phi \circ f \circ \phi^{-1}(\theta)$ as

$$\begin{aligned} &g^{(2)}(\theta) = 2\theta, \quad \theta \in [0,1) \\ &g^{(2)}(\theta) = 2\theta - 2, \quad \theta \in [1,2]. \end{aligned} \tag{2}$$

Because the map (2) has the mixing property (thus, is clearly ergodic) and preserves the Lebesgue measure $\frac{1}{2}d\theta$ of $[0,2]$, the map f preserves the measure

$$\mu(dx) = \rho(x)dx = \frac{1}{2}\frac{d\phi(x)}{dx}dx = \frac{dx}{\pi(1+x^2)}. \tag{3}$$

This is an explanation of the mechanical origin of the Cauchy distribution (3). Note that the Cauchy distribution is a simple case of Lévy stable distributions with the characteristic $\alpha = 1$. The measure (3) is absolutely continuous with respect to the Lebesgue measure, which implies that the Kolmogorov-Sinai entropy $h(\mu)$ is equivalent to the Lyapunov exponent of ln2 from the Pesin identity; and the measure is a physical one in the sense that, for almost all initial conditions x_0, the time averages $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}\delta(x - x_i)$ reproduce the invariant measure $\mu(dx)$ [10]. Our next step is to generalize exactly solvable chaos (1) to capture the full domain of Levy's stable laws. Now, let us consider the mapping

$$X_{n+1} = |\frac{1}{2}(|X_n|^\alpha - 1/|X_n|^\alpha)|^{\frac{1}{\alpha}} \cdot \mathrm{sgn}[(X_n - 1/X_n)] \equiv f_\alpha(X_n), \tag{4}$$

where $\mathrm{sgn}(x) = 1$ for $x > 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$. We prove here that this chaotic dynamics (4) also has the mixing property similar to mapping (1), as well as the exact invariant density function given by

$$\rho_\alpha(x) = \frac{\alpha}{\pi}\frac{|x|^{\alpha-1}}{(1+|x|^{2\alpha})} \simeq \frac{\alpha}{\pi}|x|^{-(\alpha+1)} \quad \text{for} \quad |x| \to \infty. \tag{5}$$

Note that the chaotic system (1) is a special case of (4) with $\alpha = 1$. We remark here that this system can also be seen as a doubling formula $s(2\theta) = f_\alpha[s(\theta)]$, where $s(\theta) = -\frac{\mathrm{sign}[\tan(\frac{\pi}{2}\theta)]}{|\tan(\frac{\pi}{2}\theta)|^{1/\alpha}}$.

Using the relations

$$\begin{array}{ll} s(2\theta) = f_\alpha[s(\theta)] & \text{for} \quad \theta \in [0,1) \\ s(2\theta - 2) = f_\alpha[s(\theta)] & \text{for} \quad \theta \in [1,2] \end{array} \tag{6}$$

and defining the diffeomorphism

$$x = \phi_\alpha^{-1}(\theta) = -\frac{\mathrm{sgn}[\tan(\frac{\pi}{2}\theta)]}{|\tan(\frac{\pi}{2}\theta)|^{1/\alpha}} \tag{7}$$

of $\theta \in [0, 2]$ into $]-\infty, +\infty[$, we obtain the piecewise-linear map $g^{(2)}(\theta) = \phi_\alpha \circ f_\alpha \circ \phi_\alpha^{-1}(\theta)$ as

$$g^{(2)}(\theta) = \begin{cases} 2\theta, & \theta \in [0,1) \\ 2\theta - 2, & \theta \in [1,2] \end{cases} \tag{8}$$

with the invariant measure $\frac{1}{2}d\theta$ of $[0, 2]$. Thus, map (4) preserves

$$\mu(dx) = \rho_\alpha(x)dx = \frac{1}{2}\frac{d\phi_\alpha}{dx}dx = \frac{\alpha}{\pi}\frac{|x|^{\alpha-1}}{(1+|x|^{2\alpha})}. \tag{9}$$

Therefore, the class of dynamical systems (4) with the parameter α also has a mixing property (thus, is ergodic) with the Lyapunov exponent ln2.

More generally, from the family of Chebyshev maps $Y_{n+1} = f(Y_n)$ defined by the addition formulas of the form $\sin^2(p\theta) = f[\sin^2(\theta)]$, where $p = 2, 3, \cdots$, with the unique density $\sigma(y) = \frac{1}{\pi\sqrt{y(1-y)}}$ of the logistic map $Y_{n+1} = 4Y_n(1-Y_n)$ (which corresponds to the case $p = 2$) [11,12], we may construct infinitely many chaotic dynamical systems $X_{n+1} = f_\alpha(X_n)$ with the unique density function $\rho_\alpha(x)$ given by Eq.(5) [13]. For example, an explicit mapping with the Lyapunov exponent ln3 is given by

$$X_{n+1} = f_\alpha(X_n) = \left| \frac{|X_n|^\alpha(|X_n|^{2\alpha} - 3)}{(3|X_n|^{2\alpha} - 1)} \right|^{\frac{1}{\alpha}} \cdot \mathrm{sign}\left[\frac{X_n(|X_n|^{2\alpha} - 3)}{(3|X_n|^{2\alpha} - 1)}\right], \tag{10}$$

which has the density (5) from the triplication formula of $s(\theta)$.

In this case, the topological conjugacy relation $g^{(3)}(\theta) = \phi_\alpha \circ f_\alpha \circ \phi_\alpha^{-1}(\theta)$ yields the piecewise-linear map:

$$g^{(3)}(\theta) = \begin{cases} 3\theta, & \theta \in [0, \frac{2}{3}) \\ 3\theta - 2, & \theta \in [\frac{2}{3}, \frac{4}{3}) \\ 3\theta - 4, & \theta \in [\frac{4}{3}, 2]. \end{cases} \tag{11}$$

In general, the same kind of topological conjugacy relation $g^{(p)}(\theta) = \phi_\alpha \circ f_\alpha \circ \phi_\alpha^{-1}(\theta)$ holds for a p-to-one piece-wise linear mapping $g^{(p)}(\theta)$. Let us consider slightly modified dynamical

systems $X_{n+1} = f_{\alpha,\delta}(X_n) \equiv \frac{1}{\delta} f_\alpha(\delta X_n)$ with a change of variable $h(x) \equiv \delta x$ for a constant $\delta > 0$. Thus, this modified dynamics,

$$f_{\alpha,\delta}(X) = |\frac{1}{2}(|X|^\alpha - 1/|\delta^2 X|^\alpha)|^{\frac{1}{\alpha}} \cdot \mathrm{sgn}[X - \frac{1}{\delta^2 X}], \tag{12}$$

has an invariant measure $\rho_{\alpha,\delta}(x)dx = \delta\rho_\alpha(\delta x)dx = \frac{\alpha\delta^\alpha |x|^{\alpha-1}dx}{\pi(1+\delta^{2\alpha}|x|^{2\alpha})}$ with a slightly modified power-law tail as

$$\rho_{\alpha,\delta}(x) \simeq \frac{\alpha}{\pi\delta^\alpha}\frac{1}{|x|^{\alpha+1}} \quad \text{for} \quad x \to \pm\infty. \tag{13}$$

We will show that this power-law tail of density is sufficient for generating arbitrary symmetric stable laws. The canonical representation of stable laws obtained by Lévy and Khintchine [14,15] is:

$$P(x;\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(izx)\psi(z)dz, \tag{14}$$

where the characteristic function $\psi(z)$ is given by

$$\psi(z) = \exp\{-i\gamma z - \eta|z|^\alpha[1 + i\beta\mathrm{sgn}(z)\omega(z,\alpha)]\}, \tag{15}$$

α, β, γ and η being real constants satisfying $0 < \alpha \le 2, -1 \le \beta \le 1, \gamma \ge 0$ and

$$\begin{array}{l} \omega(z,\alpha) = \tan(\pi\alpha/2) \quad \text{for} \quad \alpha \neq 1, \\ \omega(z,\alpha) = (2/\pi)\log|z| \quad \text{for} \quad \alpha = 1. \end{array} \tag{16}$$

According to the generalized central limit theorem (GCLT), it is known [16] that if the density function of a stochastic process has a long tail,

$$\rho(x) \simeq c_-|x|^{-(1+\alpha)} \quad \mathit{for} \quad x \to -\infty, \quad \rho(x) \simeq c_+|x|^{-(1+\alpha)} \quad \mathit{for} \quad x \to +\infty, \tag{17}$$

then the superposition $S_N = (\sum_{i=1}^N X(i) - A_N)/B_N$ of independent, identically distributed random variables $X(1), \cdots, X(N)$ with the density $\rho(x)$ converges in density to a Lévy's stable law $P(x;\alpha,\beta)$ with

$$\begin{array}{l} \beta = (c_+ - c_-)/(c_+ + c_-), \\ A_N = 0, B_N = N^{1/\alpha}, \eta = \frac{\pi(c_+ + c_-)}{2\alpha\sin(\pi\alpha/2)\Gamma(\alpha)}, \quad \text{for} \quad 0 < \alpha < 1, \\ A_N = N < x >, B_N = N^{1/\alpha}, \eta = \frac{\pi(c_+ + c_-)}{2\alpha^2\sin(\pi\alpha/2)\Gamma(\alpha-1)}, \quad \text{for} \quad 1 < \alpha < 2. \end{array} \tag{18}$$

In the case of the symmetric long-tail (5), $c_+ = c_- = \frac{\alpha}{\pi\delta^\alpha}$,$\beta = 0$ and η is determined by

$$\begin{array}{l} \eta = \frac{1}{\delta^\alpha \sin(\frac{\pi\alpha}{2})\Gamma(\alpha)} \quad \text{for} \quad 0 < \alpha < 1, \\ \eta = \frac{1}{\alpha\delta^\alpha \sin(\frac{\pi\alpha}{2})\Gamma(\alpha-1)} \quad \text{for} \quad 1 < \alpha < 2. \end{array} \tag{19}$$

Thus, according to the GCLT, the superposition of statistically independent, identically distributed random variables generated by N unique chaotic systems (4) is guaranteed to converge in distribution to an arbitrary symmetric Lévy's stable law $P(x;\alpha,\beta = 0)$. Figure

1 shows that the convergence in distribution to a Lévy's stable distribution with parameters $\alpha = 1.5$ and $\beta = 0$ is clearly seen for $N = 10000$, as predicted by the GCLT. What about more general stable distributions including asymmetric stable laws? In the next part, we will also provide chaotic dynamical systems with explicit broad distributions, whose superpositions converge in distribution to arbitrary asymmetric Lévy's stable laws. Let us consider a family of dynamical systems $X_{n+1} = f_{\alpha,\delta_1,\delta_2}(X_n)$, where

$$f_{\alpha,\delta_1,\delta_2}(X) = \begin{cases} \frac{1}{\delta_1^2|X|}(\frac{|\delta_1 X|^{2\alpha}-1}{2})^{\frac{1}{\alpha}} & \text{for} \quad X > \frac{1}{\delta_1} \\ -\frac{1}{\delta_1\delta_2|X|}(\frac{1-|\delta_1 X|^{2\alpha}}{2})^{\frac{1}{\alpha}} & \text{for} \quad 0 < X < \frac{1}{\delta_1} \\ \frac{1}{\delta_1\delta_2|X|}(\frac{1-|\delta_2 X|^{2\alpha}}{2})^{\frac{1}{\alpha}} & \text{for} \quad -\frac{1}{\delta_2} < X < 0 \\ -\frac{1}{\delta_2^2|X|}(\frac{|\delta_2 X|^{2\alpha}-1}{2})^{\frac{1}{\alpha}} & \text{for} \quad X < -\frac{1}{\delta_2}. \end{cases} \tag{20}$$

We can show that the dynamical system $X_{i+1} = f_{\alpha,\delta_1,\delta_2}(X_i)$ has an asymmetric invariant measure, $\mu(dx) = \rho(x;\alpha,\delta_1,\delta_2)dx$, where $\rho_{\alpha,\delta_1,\delta_2}(x)$ is given by

$$\rho_{\alpha,\delta_1,\delta_2}(x) = \frac{\alpha\delta_1^\alpha x^{\alpha-1}}{\pi(1+\delta_1^{2\alpha}x^{2\alpha})} \quad \text{for} \quad x > 0, \quad \rho_{\alpha,\delta_1,\delta_2}(x) = \frac{\alpha\delta_2^\alpha|x|^{\alpha-1}}{\pi(1+\delta_2^{2\alpha}|x|^{2\alpha})} \quad \text{for} \quad x < 0. \tag{21}$$

Because the power-law tail is asymmetric as

$$\rho_{\alpha,\delta_1,\delta_2}(x) \simeq \frac{\alpha}{\pi\delta_1^\alpha x^{\alpha+1}} \quad x \to +\infty, \quad \rho_{\alpha,\delta_1,\delta_2}(x) \simeq \frac{\alpha}{\pi\delta_2^\alpha |x|^{\alpha+1}} \quad x \to -\infty \tag{22}$$

for $\delta_1 \neq \delta_2$, the GCLT guarantees that the limiting distribution would be a Lévy's canonical form $P(x;\alpha,\beta)$ with the skewness parameter $\beta = \frac{\delta_2^\alpha - \delta_1^\alpha}{\delta_1^\alpha + \delta_2^\alpha} \neq 0$. Thus, we can generate arbitrary Lévy's stable laws $P(x;\alpha,\beta)$ for $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$ [17] using the chaotic mappings $f_{\alpha,\delta_1,\delta_2}(X)$ with proper parameters α, δ_1 and δ_2. Convergence to an asymmetric Lévy's stable distribution with indices $\alpha = 1.5$ and $\beta = -\frac{9-4\sqrt{2}}{7}$ is clearly seen for $N = 10000$ in Fig.2, as predicted by the GCLT. To show the exactness of the asymmetric density (21), we must check that the invariant measure $\rho_{\alpha,\delta_1,\delta_2}(x)dx$ satisfies the probability preservation relation (Perron-Frobenius Equation) [18]:

$$\rho_{\alpha,\delta_1,\delta_2}(y) = \sum_{x=f^{-1}_{\alpha,\delta_1,\delta_2}(y)} \rho_{\alpha,\delta_1,\delta_2}(x).|\frac{dx}{dy}|. \tag{23}$$

We note that $f_{\alpha,\delta,\delta}(x) = f_{\alpha,\delta}(x)$ and $\rho_{\alpha,\delta_1,\delta_2}(x) = \rho_{\alpha,\delta_1}(x)$ for $x > 0$ and $\rho_{\alpha,\delta_1,\delta_2}(x) = \rho_{\alpha,\delta_2}(x)$ for $x < 0$, which also have the Perron-Frobenius equations

$$\rho_{\alpha,\delta_i}(y) = \sum_{x=f^{-1}_{\alpha,\delta_i}(y)} \rho_{\alpha,\delta_i}(x)|\frac{dx}{dy}| \quad \text{for} \quad i = 1, 2. \tag{24}$$

Here, we define two pre-images $x_a = f^{-1}_{\alpha,\delta_1,\delta_2}(y) < 0$ and $x_b = f^{-1}_{\alpha,\delta_1,\delta_2}(y) > 0$ for $y > 0$. In the case $f_{\alpha,\delta_1}(x) = y > 0$, we also define two pre-images $x'_a = f^{-1}_{\alpha,\delta_1}(y) < 0$ and $x'_b = f^{-1}_{\alpha,\delta_1}(y)(= x_b) > 0$ for $y = f_{\alpha,\delta_1}(x) > 0$. It is easy to check that $\delta_2 x_a = \delta_1 x'_a$. From the Perron-Frobenius equations (23) and (24), we have the relation

$$\rho_{\alpha,\delta_2}(x_a)\frac{1}{|\frac{df_{\alpha,\delta_1,\delta_2}(x)}{dx}|_{x=x_a}} = \rho_{\alpha,\delta_1}(x'_a)\frac{1}{|\frac{df_{\alpha,\delta_1}(x)}{dx}|_{x=x'_a}}. \tag{25}$$

However, we can check the relation (25) under the condition $\delta_2 x_a = \delta_1 x'_a$. In the same manner, we can also show that $\rho_{\alpha,\delta_1,\delta_2}(y)$ satisfies the Perron-Frobenius equation (23) for $y < 0$.

There is also an interesting dualistic structure in these types of chaotic dynamical systems. Let us consider dynamical systems $X^*_{n+1} = f^*_{\alpha,\delta_1,\delta_2}(X^*_n)$ defined as

$$f^*_{\alpha,\delta_1,\delta_2}(X^*) = \begin{cases} -\frac{\delta_1}{\delta_2}(\frac{2|X^*|^\alpha}{|\delta_1 X^*|^{2\alpha}-1})^{\frac{1}{\alpha}} & \text{for} \quad X^* > \frac{1}{\delta_1} \\ (\frac{2|X^*|^\alpha}{1-|\delta_1 X^*|^{2\alpha}})^{\frac{1}{\alpha}} & \text{for} \quad 0 < X^* < \frac{1}{\delta_1} \\ -(\frac{2|X^*|^\alpha}{1-|\delta_2 X^*|^{2\alpha}})^{\frac{1}{\alpha}} & \text{for} \quad -\frac{1}{\delta_2} < X^* < 0 \\ \frac{\delta_2}{\delta_1}(\frac{2|X^*|^\alpha}{|\delta_2 X^*|^{2\alpha}-1})^{\frac{1}{\alpha}} & \text{for} \quad X^* < -\frac{1}{\delta_2}. \end{cases} \tag{26}$$

Because the normalized and symmetric case, $f^*_\alpha(X^*) \equiv f^*_{\alpha,\delta_1=1,\delta_2=1}(X^*)$, can also be seen as the doubling formula of $-\text{sgn}[\tan(\frac{\pi}{2}\theta)]|\tan(\frac{\pi}{2}\theta)|^{1/\alpha}$, and we can derive the piecewise-linear map $g^{(2)}(\theta) = \phi^*_\alpha \circ f^*_\alpha \circ \phi^{*-1}_\alpha(\theta)$ (2) by the diffeomorphism $x^* = \phi^{*-1}_\alpha(\theta) = -\text{sgn}[\tan(\frac{\pi}{2}\theta)]|\tan(\frac{\pi}{2}\theta)|^{1/\alpha}$, map $f^*_\alpha(X^*)$ also has the same invariant measure $\rho_\alpha(x)$ given by Eq. (5). Thus, in the same way as that used in obtaining $\rho_{\alpha,\delta_1,\delta_2}(x)$ for $f_{\alpha,\delta_1,\delta_2}(X)$, we can show that map (26) has the invariant measure $\rho_{\alpha,1/\delta_1,1/\delta_2}(x)dx$. Furthermore, the dynamical reciprocal relation

$$f^*_{\alpha,1/\delta_1,1/\delta_2}(X^*)f_{\alpha,\delta_1,\delta_2}(X) = 1 \quad \text{for} \quad XX^* = 1 \tag{27}$$

holds. This dualistic structure of the dynamical systems $f_{\alpha,\delta_1,\delta_2}(X)$ and $f^*_{\alpha,1/\delta_1,1/\delta_2}(X^*)$ originates from the relation $\phi^{-1}_\alpha(\theta) \cdot \phi^{*-1}_\alpha(\theta) = 1$.

In summary, we have found that Lévy's stable laws can be directly generated by the superposition of many independent, identically distributed dynamical variables obeying certain unique chaotic processes. Owing to the ubiquitous character of Lévy's stable laws, our methods related to ergodic transformations with long-tail densities have a broad range of applications to many different physical problems where stable laws play an essential role. This work is supported by RIKEN-SPRF and RIKEN-PSR grants.

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[17] When $\beta = \pm 1$, these cases correspond to the purely asymmetric stable distributions over the semi-infinite support $I_+ = (0, \infty)$ for $\beta = 1$ and $I_- = (-\infty, 0)$ for $\beta = -1$. These completely asymmetric stable distributions can be generated by the chaotic systems $X_{n+1} = |f_{\alpha,\delta}(X_n)|$ with the exact densities $\rho_\alpha^+(x) = \frac{2\alpha}{\pi}\frac{x^{\alpha-1}}{1+x^{2\alpha}}$ on I_+ for $\beta = 1$ and $X_{n+1} = -|f_{\alpha,\delta}(X_n)|$ with the density $\rho_\alpha^-(x) = \frac{2\alpha}{\pi}\frac{|x|^{\alpha-1}}{1+|x|^{2\alpha}}$ on I_- for $\beta = -1$.

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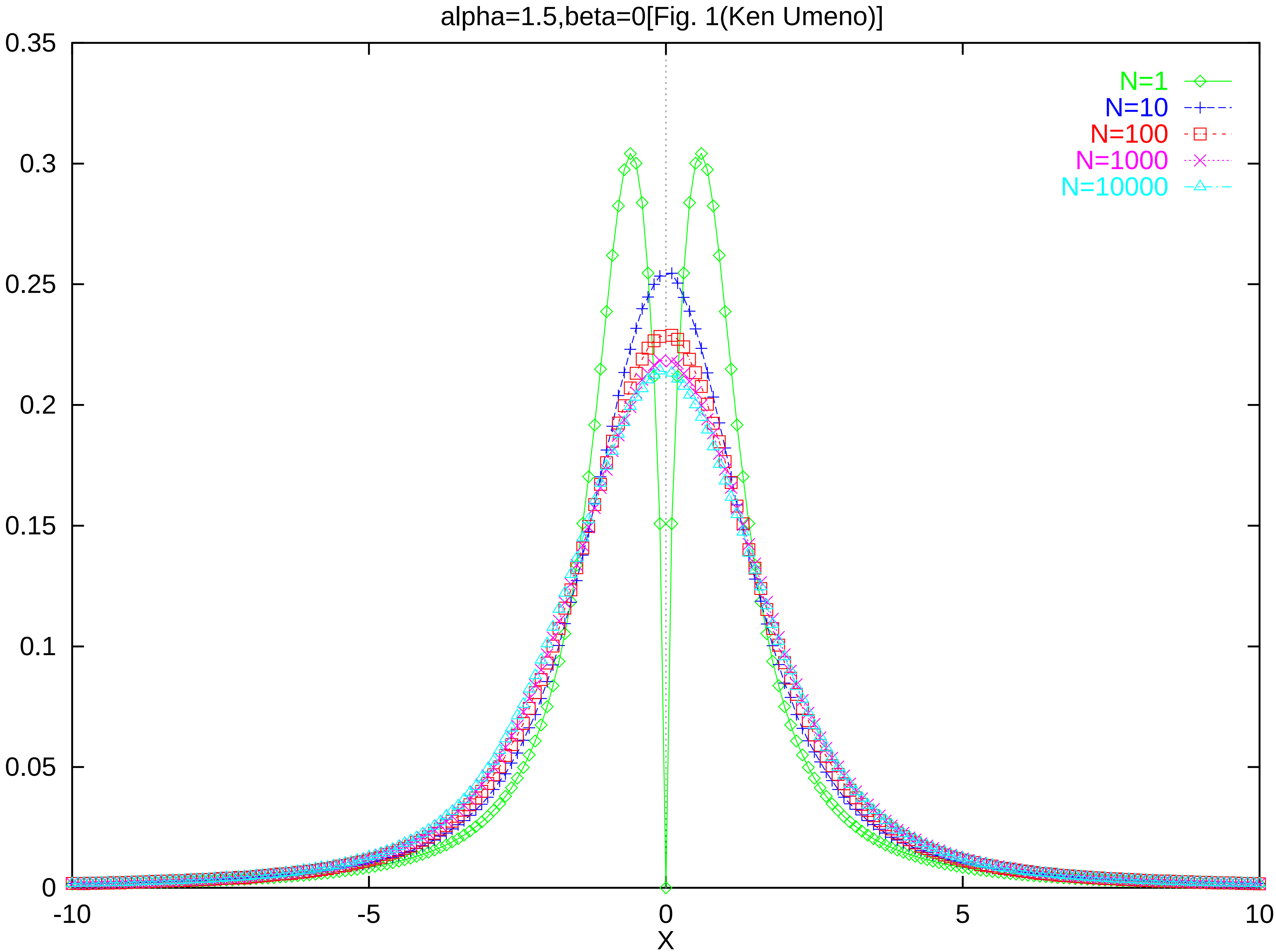
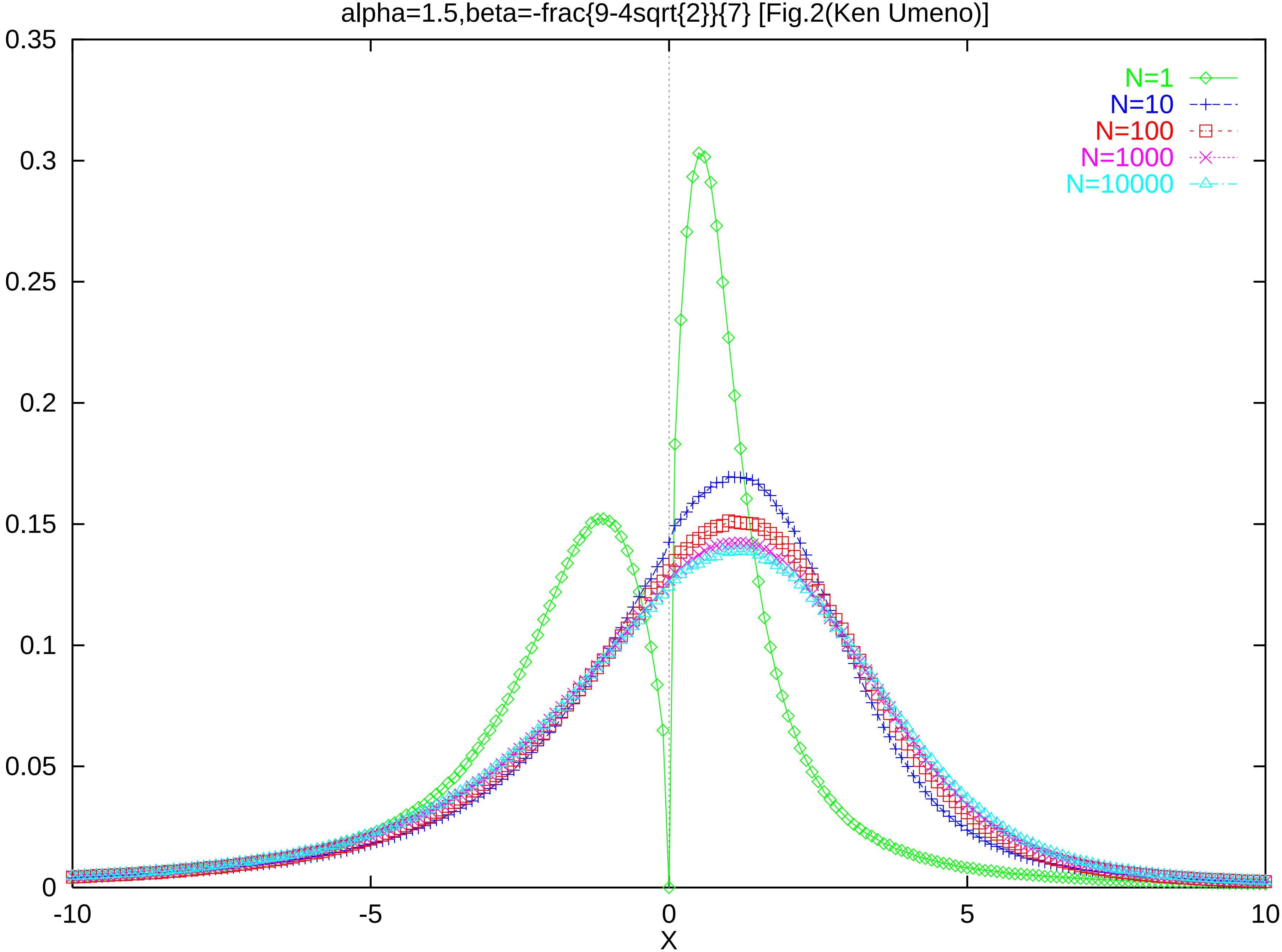
FIGURES

FIG. 1.(color) Densities of the superposition $S_N = (\sum_{i=1}^{N} X(i) - A_N)/B_N$ of dynamical variables $X(i)$ generated by chaotic systems $X_{j+1}(i) = f^{(2)}_{\alpha=1.5}[X_j(i)]$, with N different initial conditions $X_0(i)|_{i=1,\cdots,N}$ are plotted for $N = 1, 10, 100, 1000$ and 10000. In this case, the limit density converges to the symmetric Lévy's stable law with the indices $\alpha = 1.5$ and $\beta = 0$.

FIG. 2.(color) Densities of the superposition $S_N = (\sum_{i=1}^{N} X(i) - A_N)/B_N$ of dynamical variables $X(i)$ generated by chaotic systems $X_{j+1}(i) = f^{(2)}_{\alpha=1.5,\delta_1=1,\delta_2=0.5}[X_j(i)]$, with N different initial conditions $X_0(i)|_{i=1,\cdots,N}$ are plotted for $N = 1, 10, 100, 1000$ and 10000. In this case, the limit density converges to the asymmetric Lévy's stable law with the indices $\alpha = 1.5$ and $\beta = -\frac{9-4\sqrt{2}}{7}$.



alpha=1.5,beta=0[Fig. 1(Ken Umeno)]
N=1
N=10
N=100
N=1000
N=10000
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-10
-5
0
5
10
X


alpha=1.5,beta=-frac{9-4sqrt{2}}{7} [Fig.2(Ken Umeno)]
N=1
N=10
N=100
N=1000
N=10000
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-10
-5
0
5
10
X